

Transformation formulas for supercongruences

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ABSTRACT. Let p be a prime greater than 3. In the paper we prove some transformation formulae for supercongruences modulo p and solve some conjectures of Z. W. Sun.

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1. Introduction.

Let $[x]$ be the greatest integer not exceeding x , and let $\left(\frac{a}{p}\right)$ be the Legendre symbol. For a prime p let \mathbb{Z}_p be the set of rational numbers whose denominator is not divisible by p . For positive integers a, b and n , if $n = ax^2 + by^2$ for some integers x and y , we briefly say that $n = ax^2 + by^2$.

Let $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for any positive integer k . Then $\frac{(a)_k}{k!} = (-1)^k \binom{-a}{k}$. A formula of Bailey states that (see [GZ, (9) and (12)])

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} x^k &= \frac{2}{\sqrt{4-x}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}{k!^3} \left(\frac{27x^2}{(4-x)^3}\right)^k \\ &= \frac{1}{\sqrt{1-4x}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}{k!^3} \left(\frac{27x}{(4x-1)^3}\right)^k. \end{aligned}$$

It is easily seen that

$$(1.1) \quad \frac{\left(\frac{1}{2}\right)_k}{k!} = \frac{\binom{2k}{k}}{4^k} \quad \text{and} \quad \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}{k!^3} = \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{12^{3k}}.$$

Thus, taking $x = 64/m$ in Bailey's transformation and applying (1.1) we get

$$(1.2) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{m^k} = \sqrt{\frac{m}{m-16}} \sum_{k=0}^{\infty} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m}{(m-16)^3} \right)^k \\ = \sqrt{\frac{m}{m-256}} \sum_{k=0}^{\infty} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m^2}{(256-m)^3} \right)^k.$$

In Section 2, using some results in [S2, S3] we prove the following p -analogue of (1.2):

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} \equiv \left(\frac{m(m-16)}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m}{(m-16)^3} \right)^k \\ \equiv \left(\frac{m(m-256)}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m^2}{(256-m)^3} \right)^k \pmod{p},$$

where p is an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv 0, 16, 256 \pmod{p}$.

For any nonnegative integer n , we define

$$(1.3) \quad A_n = \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \binom{n}{k}^2 \quad \text{and} \quad a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}.$$

Let p be an odd prime and $u \in \mathbb{Z}_p$. In the paper we prove that for $u \not\equiv \frac{1}{4} \pmod{p}$,

$$\sum_{k=0}^{p-1} A_k u^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{u^2}{(1-4u)^3} \right)^k \pmod{p},$$

and for $u \not\equiv -1, -\frac{1}{3} \pmod{p}$,

$$\sum_{k=0}^{p-1} \binom{2k}{k}^{a_k} \left(\frac{u}{9(1+u)^2} \right)^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u}{9(1+3u)^4} \right)^k \pmod{p}.$$

As applications we prove some supercongruences involving $\{A_n\}$ or $\{a_n\}$, which were conjectured by Z.W. Sun in [Su3]. For example, if $p \equiv 1, 4 \pmod{15}$ is a prime and so $p = x^2 + 15y^2$, then

$$\sum_{n=0}^{p-1} A_n \equiv 4x^2 \pmod{p}.$$

In the paper we also pose some conjectures on supercongruences.

2. Transformation formulas involving $\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$.

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Then clearly $P_n(-x) = (-1)^n P_n(x)$. In [S1, Theorems 3.1 and 4.1] the author showed that for any prime $p > 3$ and $t \in \mathbb{Z}_p$,

$$(2.1) \quad P_{[\frac{p}{3}]}(t) \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \pmod{p}$$

and

$$(2.2) \quad \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{1-t^2}{108} \right)^k \equiv P_{[\frac{p}{3}]}(t)^2 \pmod{p}.$$

In [S2, Theorems 2.1 and 4.2] the author showed that for any prime $p > 3$ and $t \in \mathbb{Z}_p$,

$$(2.3) \quad P_{[\frac{p}{4}]}(t) \equiv -\left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \pmod{p}.$$

and

$$(2.4) \quad \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{1-t^2}{256} \right)^k \equiv P_{[\frac{p}{4}]}(t)^2 \pmod{p}.$$

In [S3], the author showed that for any prime $p > 3$ and $m, n \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$,

$$(2.5) \quad \left(\sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right) \right)^2 \equiv \left(\frac{-3m}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{4m^3 + 27n^2}{12^3 \cdot 4m^3} \right)^k \pmod{p}.$$

Theorem 2.1. *For any prime $p > 3$ and $m \in \mathbb{Z}_p$ with $m(m-16)(m-256) \not\equiv 0 \pmod{p}$, we have*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} &\equiv \left(\frac{m(m-16)}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m}{(m-16)^3} \right)^k \\ &\equiv \left(\frac{m(m-256)}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m^2}{(256-m)^3} \right)^k \pmod{p}. \end{aligned}$$

Proof. By [S2, Theorem 3.2],

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} \equiv \left(\frac{m(m-64)}{p} \right) P_{[\frac{p}{4}]} \left(\frac{m+64}{m-64} \right)^2 \pmod{p}.$$

Thus, applying (2.3) and (2.5) we deduce that

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} &\equiv \left(\frac{m(m-64)}{p} \right) \left(\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3}{2} \left(3 \cdot \frac{m+64}{m-64} + 5 \right) x + 9 \cdot \frac{m+64}{m-64} + 7 \right)}{p} \right)^2 \\ &\equiv \left(\frac{m(m-16)}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m}{(m-16)^3} \right)^k \pmod{p}. \end{aligned}$$

As $P_n(-x) = (-1)^n P_n(x)$, we also have

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} &\equiv \left(\frac{m(m-64)}{p} \right) P_{[\frac{p}{4}]} \left(-\frac{m+64}{m-64} \right)^2 \\ &\equiv \left(\frac{m(m-64)}{p} \right) \left(\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3}{2} \left(-3 \cdot \frac{m+64}{m-64} + 5 \right) x - 9 \cdot \frac{m+64}{m-64} + 7 \right)}{p} \right)^2 \\ &\equiv \left(\frac{m(m-256)}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m^2}{(256-m)^3} \right)^k \pmod{p}. \end{aligned}$$

Conjecture 2.1. *Let $p > 3$ be a prime. Then*

(i) *If $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \equiv \left(\frac{3}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{12^{3k}} \equiv \left(\frac{33}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{66^{3k}} \pmod{p^3}.$$

(ii) *If $p \equiv 1, 2, 4 \pmod{7}$, then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \equiv \left(\frac{-15}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-15)^{3k}} \equiv \left(\frac{-255}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{255^{3k}} \pmod{p^3}.$$

(iii) *If $p \equiv 1, 3 \pmod{8}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{5}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{20^{3k}} \pmod{p^3}.$$

(iv) *If $p \equiv 1 \pmod{3}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{256^k} \equiv \left(\frac{-5}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{54000^k} \pmod{p^3}.$$

For other conjectures on $\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 / m^k \pmod{p^3}$, see [Su1, Conjectures A3, A35 and A36].

Theorem 2.2. *For any prime $p > 3$ and $t \in \mathbb{Z}_p$ with $4t \not\equiv \pm 5 \pmod{p}$, we have*

$$\begin{aligned}
& \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{1-t^2}{108} \right)^k \\
& \equiv \left(\frac{5-4t}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{(t-1)(t+1)^3}{432(4t-5)^3} \right)^k \\
& \equiv \left(\frac{5+4t}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{(t+1)(1-t)^3}{432(4t+5)^3} \right)^k \pmod{p}.
\end{aligned}$$

Proof. By (2.1), (2.2) and (2.5),

$$\begin{aligned}
& \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{1-t^2}{108} \right)^k \\
& \equiv P_{[\frac{p}{3}]}(t)^2 \equiv \left(\sum_{x=0}^{p-1} \left(\frac{x^3 + 3(4t-5)x + 2(2t^2 - 14t + 11)}{p} \right) \right)^2 \\
& \equiv \left(\frac{5-4t}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{(t-1)(t+1)^3}{432(4t-5)^3} \right)^k \pmod{p}.
\end{aligned}$$

Substituting t with $-t$ in the above we obtain the remaining result.

We remark that Theorem 2.2 is the p -analogue of the Kummer-Coursat transformation in [GZ, (20)].

Theorem 2.3. *For any prime $p > 3$ and $t \in \mathbb{Z}_p$ with $3t \not\equiv \pm 5 \pmod{p}$, we have*

$$\begin{aligned}
& \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{1-t^2}{256} \right)^k \\
& \equiv \left(\frac{10+6t}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{(t-1)^2(t+1)}{64(3t+5)^3} \right)^k \\
& \equiv \left(\frac{10-6t}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{(t+1)^2(t-1)}{64(3t-5)^3} \right)^k \pmod{p}.
\end{aligned}$$

Proof. By (2.3), (2.4) and (2.5),

$$\begin{aligned}
& \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{1-t^2}{256} \right)^k \\
& \equiv P_{[\frac{p}{4}]}(t)^2 \equiv \left(\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \right)^2 \\
& \equiv \left(\frac{10+6t}{p} \right) \sum_{k=0}^{[p/6]} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{(t-1)^2(t+1)}{64(3t+5)^3} \right)^k \pmod{p}.
\end{aligned}$$

Substituting t with $-t$ in the above we obtain the remaining result.

3. Congruences involving $\{A_n\}$.

Lemma 3.1. *For any nonnegative integer n ,*

$$\sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{n+k}{3k} 4^{n-2k} = \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} \binom{n}{k}^2.$$

Proof. Let m be a nonnegative integer. For $k \in \{0, 1, \dots, m\}$ set

$$\begin{aligned}
F_1(m, k) &= \binom{2k}{k}^2 \binom{3k}{k} \binom{m+k}{3k} 4^{m-2k}, \\
F_2(m, k) &= \binom{2k}{k} \binom{2m-2k}{m-k} \binom{m}{k}^2.
\end{aligned}$$

For $k \in \{0, 1, \dots, m+1\}$ set

$$\begin{aligned}
G_1(m, k) &= -\frac{192(3m+4)k^4}{(m+1+k)(m+2+k)} \binom{2k}{k}^2 \binom{3k}{k} \binom{m+k+2}{3k} 4^{m-2k}, \\
G_2(m, k) &= \frac{2k^3(-12m^3 - 62m^2 - 104m - 56 + 26km^2 + 89km + 74k - 18k^2m - 30k^2 + 4k^3)}{(m+2-k)^3} \\
&\quad \times \binom{2k}{k} \binom{2(m+1-k)}{m+1-k} \binom{m+1}{k}^2.
\end{aligned}$$

For $i = 1, 2$ and $k \in \{0, 1, \dots, m\}$, using Maple it is easy to check that

$$\begin{aligned}
(3.1) \quad & (m+2)^3 F_i(m+2, k) - 2(2m+3)(5m^2 + 15m + 12) F_i(m+1, k) + 64(m+1)^3 F_i(m, k) \\
& = G_i(m, k+1) - G_i(m, k).
\end{aligned}$$

Set $S_i(n) = \sum_{k=0}^n F_i(n, k)$ for $n = 0, 1, 2, \dots$. Then

$$\begin{aligned}
& (m+2)^3(S_i(m+2) - F_i(m+2, m+2) - F_i(m+2, m+1)) \\
& - 2(2m+3)(5m^2 + 15m + 12)(S_i(m+1) - F_i(m+1, m+1)) \\
& + 64(m+1)^3 S_i(m) \\
& = \sum_{k=0}^m \left((m+2)^3 F_i(m+2, k) - 2(2m+3)(5m^2 + 15m + 12) F_i(m+1, k) \right. \\
& \quad \left. + 64(m+1)^3 F_i(m, k) \right) \\
& = \sum_{k=0}^m (G_i(m, k+1) - G_i(m, k)) = G_i(m, m+1) - G_i(m, 0) \\
& = G_i(m, m+1).
\end{aligned}$$

Thus, for $i = 1, 2$ and $m = 0, 1, 2, \dots$,

$$\begin{aligned}
& (m+2)^3 S_i(m+2) - 2(2m+3)(5m^2 + 15m + 12) S_i(m+1) \\
& + 64(m+1)^3 S_i(m) \\
(3.2) \quad & = G_i(m, m+1) + (m+2)^3 (F_i(m+2, m+2) + F_i(m+2, m+1)) \\
& - 24(2m+3)(18m^2 + 54m + 41) F_i(m+1, m+1) = 0.
\end{aligned}$$

Since $S_1(0) = 1 = S_2(0)$ and $S_1(1) = 4 = S_2(1)$, from (3.2) we deduce $S_1(n) = S_2(n)$ for all $n = 0, 1, 2, \dots$. This completes the proof.

Let $\{A_n\}$ be given by (1.3). We have the following result.

Theorem 3.1. *Let p be an odd prime and $u \in \mathbb{Z}_p$ with $u \not\equiv \frac{1}{4} \pmod{p}$. Then*

$$\sum_{n=0}^{p-1} A_n u^n \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{u^2}{(1-4u)^3} \right)^k \pmod{p}.$$

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Proof. By Lemma 3.1,

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{u^2}{(1-4u)^3} \right)^k \\
& \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} u^{2k} (1-4u)^{p-1-3k} \\
& = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} u^{2k} \sum_{r=0}^{p-1-3k} \binom{p-1-3k}{r} (-4u)^r \\
& = \sum_{n=0}^{p-1} u^n \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{p-1-3k}{n-2k} (-4)^{n-2k} \\
& \equiv \sum_{n=0}^{p-1} u^n \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{-1-3k}{n-2k} (-4)^{n-2k} \\
& = \sum_{n=0}^{p-1} u^n \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{n+k}{3k} 4^{n-2k} = \sum_{n=0}^{p-1} A_n u^n \pmod{p}.
\end{aligned}$$

Thus the theorem is proved.

Remark 3.1 Using Lemma 3.1 and similar arguments in the proof of Theorem 3.1 we obtain

$$(3.3) \quad \sum_{n=0}^{\infty} A_n u^n = \frac{1}{1-4u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{3k}{k} \left(\frac{u^2}{(1-4u)^3} \right)^k,$$

where $|u|$ is sufficiently small. In [R], Rogers deduced (3.3) by using much advanced and complicated method.

Theorem 3.2. *Let p be a prime such that $p \equiv 1, 4 \pmod{5}$. Then*

$$\sum_{n=0}^{p-1} A_n \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 0 \pmod{p} & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Proof. Taking $u = 1$ in Theorem 3.1 and then applying [S1, Theorem 4.6] we obtain the result.

Conjecture 3.1. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \sum_{k=0}^n \binom{n}{k}^4 \equiv \left(\frac{p}{3} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-27)^k} \pmod{p^3}.$$

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If $p \equiv 1, 17, 19, 23 \pmod{30}$, we also have

$$\sum_{n=0}^{p-1} A_n \equiv \sum_{n=0}^{p-1} \frac{A_n}{64^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-27)^k} \pmod{p^3}.$$

Remark 3.2 In [Su3], Z.W. Sun conjectured that for any prime $p > 5$,

$$\begin{aligned} \left(\frac{p}{3}\right) \sum_{n=0}^{p-1} \sum_{k=0}^n \binom{n}{k}^4 &\equiv \sum_{n=0}^{p-1} A_n \equiv \sum_{n=0}^{p-1} \frac{A_n}{64^n} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 15y^2 \equiv 1, 4 \pmod{15}, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 5y^2 \equiv 2, 8 \pmod{15}, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases} \end{aligned}$$

Theorem 3.3. Let p be a prime such that $p \equiv 1, 7, 17, 23 \pmod{24}$. Then

$$\sum_{n=0}^{p-1} \frac{A_n}{(-8)^n} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. Taking $u = -\frac{1}{8}$ in Theorem 3.1 and then applying [S1, Theorem 4.5] we obtain the result.

Conjecture 3.2. Let p be a prime such that $p \equiv 1, 5, 7, 11 \pmod{24}$. Then

$$\sum_{n=0}^{p-1} \frac{A_n}{(-8)^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{216^k} \pmod{p^3}.$$

Remark 3.3 In [Su3], Z.W. Sun conjectured that for any prime $p > 3$,

$$\sum_{n=0}^{p-1} \frac{A_n}{(-8)^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Theorem 3.4. Let p be an odd prime. Then

$$\sum_{n=0}^{p-1} \frac{A_n}{8^n} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. Taking $u = \frac{1}{8}$ in Theorem 3.1 and then applying [S1, Theorem 4.3] we obtain the result.

Conjecture 3.3. *Let p be a prime such that $p \equiv 1, 3 \pmod{8}$. Then*

$$\sum_{n=0}^{p-1} \frac{A_n}{8^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{8^k} \pmod{p^3}.$$

Remark 3.4 In [Su3], Z.W. Sun conjectured that for any odd prime p ,

$$\sum_{n=0}^{p-1} \frac{A_n}{8^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Conjecture 3.4. *Let $p > 3$ be a prime.*

(i) *For $p \equiv 2 \pmod{3}$ we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{1458^k} \equiv 0 \pmod{p^3}.$$

(ii) *We have*

$$\sum_{n=0}^{p-1} \frac{A_n}{4^n} \equiv \sum_{n=0}^{p-1} \frac{A_n}{(-32)^n} \equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \pmod{p^3}.$$

(iii) *For $p \equiv 1 \pmod{3}$ we have*

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{A_n}{(-2)^n} &\equiv \sum_{n=0}^{p-1} \frac{A_n}{16^n} \\ &\equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{1458^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}. \end{aligned}$$

Remark 3.5 In [Su1], Z.W. Sun conjectured that for any prime $p \equiv 1 \pmod{3}$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-144)^k} \pmod{p^3}.$$

In [Su3], Z.W. Sun conjectured that for any prime $p > 3$,

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{A_n}{(-2)^n} &\equiv \sum_{n=0}^{p-1} \frac{A_n}{4^n} \equiv \sum_{n=0}^{p-1} \frac{A_n}{16^n} \equiv \sum_{n=0}^{p-1} \frac{A_n}{(-32)^n} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

4. Congruences involving $\{a_n\}$.

Let $\{a_n\}$ be given by (1.3). Using Maple we find that

$$(4.1) \quad (n+2)^2 a_{n+2} - (10n^2 + 30n + 23)a_{n+1} + 9(n+1)^2 a_n = 0 \quad (n = 0, 1, 2, \dots).$$

Lemma 4.1. *For any nonnegative integers n , we have*

$$\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} (-9)^{n-k} a_k = \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+3k}{4k} (-27)^{n-k}.$$

Proof. Let

$$S_1(n) = \sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} (-9)^{n-k} a_k,$$

$$S_2(n) = \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+3k}{4k} (-27)^{n-k}.$$

Then $S_1(0) = 1 = S_2(0)$ and $S_1(1) = -3 = S_2(1)$. Using Maple and (4.1) we find that for $i = 1, 2$ and $m = 0, 1, 2, \dots$,

$$(m+2)^3 S_i(m+2) + (2m+3)(7m^2 + 21m + 17) S_i(m+1) + 81(m+1)^3 S_i(m) = 0.$$

Thus $S_1(n) = S_2(n)$. This proves the lemma.

Theorem 4.1. *Let p be an odd prime and $u \in \mathbb{Z}_p$ with $u(u+1)(3u+1) \not\equiv 0 \pmod{p}$.*

(i) *We have*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \left(\frac{u}{9(1+u)^2} \right)^k a_k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u}{9(1+3u)^4} \right)^k \pmod{p}.$$

(ii) *If $u \not\equiv -3 \pmod{p}$, then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u}{9(1+3u)^4} \right)^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u^3}{9(3+u)^4} \right)^k \pmod{p}.$$

Proof. On the one hand,

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{2k}{k} \left(\frac{u}{9(1+u)^2} \right)^k a_k \\
& \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \frac{a_k}{9^k} u^k (1+u)^{p-1-2k} = \sum_{k=0}^{p-1} \binom{2k}{k} \frac{a_k}{9^k} u^k \sum_{r=0}^{p-1-2k} \binom{p-1-2k}{r} u^r \\
& = \sum_{k=0}^{p-1} \binom{2k}{k} \frac{a_k}{9^k} u^k \sum_{r=0}^{p-1} \binom{p-1-2k}{r} u^r \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \frac{a_k}{9^k} u^k \sum_{r=0}^{p-1} \binom{-1-2k}{r} u^r \\
& = \sum_{n=0}^{p-1} u^n \sum_{k=0}^n \binom{2k}{k} \frac{a_k}{9^k} \binom{-1-2k}{n-k} \\
& = \sum_{n=0}^{p-1} u^n \sum_{k=0}^n \binom{2k}{k} \frac{a_k}{9^k} (-1)^{n-k} \binom{n+k}{2k} \\
& = \sum_{n=0}^{p-1} \frac{u^n}{9^n} \sum_{k=0}^n \binom{2k}{k} a_k (-9)^{n-k} \binom{n+k}{2k} \pmod{p}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u}{9(1+3u)^4} \right)^k \\
& \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \frac{u^k}{9^k} (1+3u)^{p-1-4k} = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \frac{u^k}{9^k} \sum_{r=0}^{p-1} \binom{p-1-4k}{r} (3u)^r \\
& \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \frac{u^k}{9^k} \sum_{r=0}^{p-1} \binom{-1-4k}{r} (3u)^r \\
& = \sum_{n=0}^{p-1} u^n \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} \frac{1}{9^k} \binom{-1-4k}{n-k} 3^{n-k} \\
& = \sum_{n=0}^{p-1} u^n \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} 3^{n-3k} (-1)^{n-k} \binom{n+3k}{4k} \\
& = \sum_{n=0}^{p-1} \frac{u^n}{9^n} \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} (-27)^{n-k} \binom{n+3k}{4k} \pmod{p}.
\end{aligned}$$

Now combining all the above with Lemma 4.1 we deduce (i).

If $u \not\equiv -3 \pmod{p}$, substituting u with $\frac{1}{u}$ in (i) we obtain

$$\sum_{k=0}^{p-1} \binom{2k}{k} \left(\frac{u}{9(1+u)^2} \right)^k a_k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u^3}{9(3+u)^4} \right)^k \pmod{p}.$$

Thus (ii) follows.

Remark 4.1 Using Lemma 4.1 and similar arguments in the proof of Theorem 4.1 we obtain

$$(4.2) \quad \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{u}{9(1+u)^2} \right)^k a_k = \frac{1+u}{1+3u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{u}{9(1+3u)^4} \right)^k.$$

In [R], Rogers deduced (4.2) by using advanced and complicated method.

Corollary 4.1. *Let p be a prime such that $p \equiv \pm 1 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{36^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. Taking $u = 1$ in Theorem 4.1 and then applying [S2, Theorem 5.4] we obtain the result.

Conjecture 4.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{36^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Corollary 4.2. *Let $p > 7$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{100^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. From [M] and [Su2] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Now taking $u = 9$ in Theorem 4.1 and then applying the above we obtain the result.

Conjecture 4.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{100^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Remark 4.2 In [Su1], Zhi-Wei Sun conjectured that for any prime $p \neq 2, 3, 7$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Corollary 4.3. *Let p be a prime such that $p \equiv \pm 1 \pmod{12}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{(-12)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Proof. Taking $u = -3$ in Theorem 4.1(i) and then applying [S2, Theorem 5.3] we obtain the result.

Conjecture 4.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{(-12)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and so } p = x^2 + 9y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } 12 \mid p-5 \text{ and so } 2p = x^2 + 9y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

For any nonnegative integers n , define

$$b_n = \sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+3k}{4k} (-27)^{n-k}.$$

It is known that

$$b_n = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{2k}{k} \binom{3k}{k} \binom{n}{3k} \binom{n+k}{k} (-3)^{n-3k}.$$

We have the following conjectures involving $\{b_n\}$.

Conjecture 4.4. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1} b_n \equiv \sum_{n=0}^{p-1} \frac{b_n}{81^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Conjecture 4.5. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{b_n}{9^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Conjecture 4.6. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{b_n}{(-9)^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture 4.7. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{b_n}{(-3)^n} &\equiv \sum_{n=0}^{p-1} \frac{b_n}{(-27)^n} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and so } p = x^2 + 9y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } 12 \mid p-5 \text{ and so } 2p = x^2 + 9y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

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